J. Differential Equations 247 (2009) 2440–2454

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Solvability conditions for some semi-linear parabolic equations

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ARTICLE INFO

Article history:
Received 30 August 2008
Revised 24 July 2009
Available online 15 August 2009

ABSTRACT

Consider the Cauchy problem of the semi-linear parabolic equation

$$\Delta u - \partial_t u + V u^p + w = 0, \quad p > 1,$$

in $\mathbb{R}^N \times (0, T)$, $T > 0$, $u(x, 0) = u_0(x) \geqslant 0$, and $u_0, V, w \in L^1_{loc}(\mathbb{R}^N)$. We establish a necessary condition on the nonlinear potential $V = V(x)$ so that the above has a positive solution for some $w = w(x) \geqslant 0$, $u_0 = u_0(x) \geqslant 0$. An application to the case when V is the important inverse square potential is also given.

1. Introduction

The purpose of the paper is to prove that certain conditions are necessary for the existence of positive solutions to the Cauchy problem

$$\begin{cases} \Delta u(x,t) - \partial_t u(x,t) + V(x) u^p(x,t) + w(x) = 0, \quad p > 1, \\ u(x,t) = u_0(x). \end{cases} \tag{1.1}$$

Here $\Delta u = \sum_{i=1}^N u_{x_i x_i}$ and $V = V(x)$, $w = w(x)$, $u_0 = u_0(x) \geqslant 0$ with $V, w \in L^1_{loc}(\mathbb{R}^N)$; $x \in \mathbb{R}^N$ and $t > 0$.

For many years, the equation, being in the core of nonlinear analysis, has been studied by many authors. There are many sufficient conditions on the functions V, w and u_0 such that (1.1) has positive solutions. See e.g. [5]. However, as far as we know, the necessity is missing with the exception of [3]. There, Baras and Pierre introduced an interesting implicit necessary condition. A well-known open

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2009.07.028

problem that has been around for a long time is to find an explicit solvability condition on V, w and u_0. The main goal of the paper is to answer this question for a wide class of V and some w, u_0, the nonlinear potential in the equation. See Theorem 3.1 which is the main theorem of the paper.

The elliptic counterpart of the problem was solved by Kalton and Verbitsky in the important paper [4]; and also by Brezis and Cabré, in another important paper [1]. In [1], the authors also studied the parabolic case when V is the important inverse square potential. Specifically, they proved that when $V = 1/|x|^2$ and $p = 2$, then (1.1) has no positive solutions. However, the case when $p \neq 2$ was left open. As an application of our main theorem, it will be shown at the end of Section 3 that this nonexistence result actually holds for all $p > 1$ (cf. Corollary 3.2).

The proof of our results follows the general road map developed in [4] for the elliptic case. But the original ideas there are very dependent on the quasi-metric assumption for the Green function K of the Laplacian. Recall that K is said to satisfy a quasi-metric condition if

$$\exists \kappa > 0 \quad \text{s.t.} \quad \frac{1}{K(x,y)} \leqslant \kappa \left(\frac{1}{K(x,z)} + \frac{1}{K(y,z)} \right).$$

This condition is false for the heat kernel. We need to develop a scheme using shifts in the time direction to manufacture what is accomplished by the quasi-metric property. We will prove the main necessity condition for the nonlinear potential V in Section 3.

At this time we are unable to find a similar solvability condition for the inhomogeneous term w, as was done in [4] for the elliptic case. We hope to address this problem in the future.

We now fix some notation and conventions. We denote the fundamental solution to the heat equation as

$$G(x,t;y,s) = \frac{1}{(4\pi(t-s))^{N/2}} e^{\frac{-|x-y|^2}{4(t-s)}}, \quad t > s.$$

Definition 1.1. An integrable function u is called a solution to (1.1) in $\mathbb{R}^N \times (0,T]$ if

$$u(x,t) = \int G(x,t;y,0)u_0(y)\,dy + \int_0^t\!\!\int G(x,t;y,s)V(y)u^p(y,s)\,dy\,ds$$
$$+ \int_0^t\!\!\int G(x,t;y,s)w(y)\,dy\,ds$$

for all $(x,t) \in \mathbb{R}^N \times (0,T]$.

If also $u(x,t) > 0$ a.e. for $t > 0$, then u is called a positive solution to (1.1).

Solutions of this kind are sometimes referred to as mild solutions. Note that we do not presume the solution has any smoothness.

Frequently, our calculations will involve the heat ball $P(x,t;r)$ defined by

$$P(x,t;r) = \left\{ (y,s)\colon\ G(x,t;y,s) \geqslant \frac{1}{r} \right\}.$$

We think of the point (x,t) as the "center" or vortex and r as the radius. By definition the heat ball $P(x,t;r)$ is oval shaped. One can see in terms of relative heat ball size that if $r_2 > r_1$, then $P(x,t,r_1) \subset P(x,t,r_2)$. Furthermore, it can be shown that the radius of the widest cross section is $\sqrt{\frac{N}{2\pi e}} r^{1/N}$. Also the largest interval in time contained in the ball has length $r^{2/N}/(4\pi)$.

The calculations often involve weighted convolutions. The weight will typically be the potential term V in (1.1).

Definition 1.2. Given $g \in L^1_{loc}(\mathbb{R}^N \times (0,\infty))$, we define the weighted convolution with the heat kernel G under a weight function $f \in L^1_{loc}(\mathbb{R}^N)$ as

$$(G *_f g)(x,t) = \int_0^t\!\!\int G(x,t;y,s) f(y) g(y,s)\,dy\,ds.$$

when the right-hand side is finite a.e.

In Section 2 we will prove a result on the geometry on heat balls which will replace the quasi-metric assumption in the elliptic case. The main theorem and its proof will be given in Section 3.

2. A containment property of heat balls

In this section we state and prove a result on the geometry of heat balls, which will be essential in proving the main results. The proof is an elementary but tedious calculation involving the triangle inequality.

Proposition 2.1. *Let P be the parabolic ball $P(x,t;a)$ for $a > 0$ and $P_j = P(x,t;\frac{a}{2^j})$ for $j \in \mathbb{N}$. Then there exists a dimensional constant $b = b(N) > 0$ and a positive integer σ such that*

$$P\left(x, t - \lambda\left(2^j\right)^{-2/N}; \frac{a}{2^j}\right) \subset P(x,t;a) \cap P\left(y, s; \frac{ba}{2^{j+1}}\right), \tag{2.1}$$

for all $(y,s) \in P_j$ with $j \geqslant \sigma$. Here $\lambda = a^{2/N}/(2\pi)$.

Proof. First let us prove that, for all nonnegative integers j,

$$P\left(x, t - \lambda\left(2^j\right)^{-2/N}; \frac{a}{2^j}\right) \subset P\left(y, s; \frac{ba}{2^{j+1}}\right).$$

By scaling invariance, we can and do take $j = 0$.

We want to prove, for $(y,s) \in P(x,t;a)$ and $(z,\tau) \in P(x,t-\lambda;a)$, it holds $(z,\tau) \in P(y,s;ba/2)$ for $b = b(N)$. Since we have $(y,s) \in P(x,t;a)$, by definition

$$\frac{1}{(4\pi(t-s))^{N/2}} e^{\frac{-|x-y|^2}{4(t-s)}} \geqslant \frac{1}{a}. \tag{2.2}$$

Similarly since $(z,\tau) \in P(x,t-\lambda;a)$

$$\frac{1}{(4\pi(t-\lambda-\tau))^{N/2}} e^{\frac{-|x-z|^2}{4(t-\lambda-\tau)}} \geqslant \frac{1}{a}. \tag{2.3}$$

Applying ln to both sides of (2.2) gives

$$\frac{-|x-y|^2}{4(t-s)} \geqslant -\ln\left(\frac{a}{(4\pi(t-s))^{N/2}}\right),$$

$$|x-y|^2 \leqslant \big[2N(t-s)\big]\ln\left(\frac{a^{2/N}}{4\pi(t-s)}\right). \tag{2.4}$$

The same calculation can be made for (2.3) to give

$$|x-z|^2 \leqslant 2N(t-\lambda-\tau)\ln\left(\frac{a^{2/N}}{4\pi(t-\lambda-\tau)}\right). \tag{2.5}$$

We want to prove $(z,\tau)\in P(y,s;ab/2)$, which according to the above calculations is equivalent to having

$$|y-z|^2 \leqslant 2N(s-\tau)\ln\left(\frac{(ab/2)^{2/N}}{4\pi(s-\tau)}\right). \tag{2.6}$$

But from the triangle inequality we see that

$$|y-z| \leqslant |x-y|+|x-z|. \tag{2.7}$$

Hence we have

$$|y-z|^2 \leqslant 2|x-y|^2+2|x-z|^2. \tag{2.8}$$

We have already obtained bounds on $|x-y|^2$ and $|x-z|^2$ in (2.4) and (2.5) respectively. Hence from using (2.4) and (2.5) with (2.8) we obtain

$$\begin{aligned}|y-z|^2 &\leqslant \big[4N(t-s)\big]\ln\left(\frac{a^{2/N}}{4\pi(t-s)}\right)+\big[4N(t-\lambda-\tau)\big]\ln\left(\frac{a^{2/N}}{4\pi(t-\lambda-\tau)}\right)\\ &= 2N(s-\tau)\left\{\left[\frac{2(t-s)}{s-\tau}\ln\left(\frac{a^{2/N}}{4\pi(t-s)}\right)\right]+\frac{2(t-\lambda-\tau)}{s-\tau}\ln\left(\frac{a^{2/N}}{4\pi(t-\lambda-\tau)}\right)\right\}.\end{aligned}$$

But we can dominate this by maximizing the function $f(m)=m\ln(\frac{a^{2/N}}{4\pi m})$ where $m>0$. Note that the maximum value of f is $a^{2/N}/(4\pi e)$ which is reached when $m=a^{2/N}/(4\pi e)$. Hence we have

$$\begin{aligned}|y-z|^2 &\leqslant 2N(s-\tau)\left[\frac{2}{s-\tau}\left(\frac{a^{2/N}}{4\pi e}\right)+\frac{2}{s-\tau}\left(\frac{a^{2/N}}{4\pi e}\right)\right]\\ &\leqslant 2N(s-\tau)\left[\left(\frac{1}{\lambda-(\frac{a^{2/N}}{4\pi})}\right)\left(\frac{a^{2/N}}{\pi e}\right)\right]\\ &= 2N(s-\tau)\left(\frac{a^{2/N}}{\pi e}\right)\left(\frac{1}{\lambda-(\frac{a^{2/N}}{4\pi})}\right).\end{aligned} \tag{2.9}$$

The second to last step is justified by the following arguments. Since $(y,s)\in P(x,t;a)$ we have $0<t-s<a^{2/N}/(4\pi)$ and hence

$$s>t-\big(a^{2/N}/(4\pi)\big). \tag{2.10}$$

Similarly since we have $(z,\tau)\in P(x,t-\lambda;a)$ we know that $0<t-\tau-\lambda<a^{2/N}/(4\pi)$ which implies

$$\tau<t-\lambda. \tag{2.11}$$

Therefore computing $s-\tau$ from (2.10) and (2.11) directly gives

$$\frac{1}{s-\tau}\leqslant\frac{1}{\lambda-(a^{2/N}/(4\pi))},$$

which justifies the previous estimate.

As the minimum of τ is $t-(\frac{a^{2/N}}{4\pi})-\lambda$; and the maximum of s is t we have

$$s-\tau\leqslant\left(\frac{a^{2/N}}{4\pi}\right)+\lambda=\frac{3}{4\pi}a^{2/N},$$

since $\lambda=\frac{a^{2/N}}{2\pi}$. With this we can compute

$$\ln\left[\frac{(ab/2)^{2/N}}{4\pi(s-\tau)}\right]\geqslant\ln\left[\frac{(ab/2)^{2/N}}{4\pi}\left(\frac{4\pi}{3}\right)\left(\frac{1}{a}\right)^{2/N}\right]=\ln\left[\frac{(b/2)^{2/N}}{3}\right]. \tag{2.12}$$

Now all we choose b such that

$$\ln\left[\frac{(b/2)^{2/N}}{3}\right]=\left(\frac{a^{2/N}}{\pi e}\right)\left(\frac{1}{\lambda-(\frac{a^{2/N}}{4\pi})}\right)=\left(\frac{a^{2/N}}{\pi e}\right)\left(\frac{1}{(\frac{a^{2/N}}{2\pi})-(\frac{a^{2/N}}{4\pi})}\right)=\frac{4}{e},$$

i.e. $b=2(3e^{4/e})^{N/2}$.

So that in totality we have, from (2.9) and (2.12)

$$\begin{aligned}|y-z|^2&\leqslant 2N(s-\tau)\left(\frac{a^{2/N}}{\pi e}\right)\left(\frac{1}{\lambda-(\frac{a^{2/N}}{4\pi})}\right)\\&=2N(s-\tau)\ln\left[\frac{(b/2)^{2/N}}{3}\right]\\&\leqslant 2N(s-\tau)\ln\left[\frac{(ab/2)^{2/N}}{4\pi(s-\tau)}\right].\end{aligned}$$

This is (2.6). Hence we see that indeed $(z,\tau)\in P(y,s;ab/2)$, showing that $P(x,t-\lambda(2^j)^{-2/N};\frac{a}{2^j})\subset P(y,s;\frac{ba}{2^{j+1}})$ after scaling.

Next we prove

$$P\left(x,t-\lambda\left(2^j\right)^{-2/N};\frac{a}{2^j}\right)\subset P(x,t;a)$$

when $j\geqslant\sigma>0$. Let $(z,\tau)\in P(x,t-\lambda(2^j)^{-2/N};\frac{a}{2^j})$. The containment follows if we can prove the second inequality in the following expression:

$$|x-z|^2\leqslant 2N\left(t-\lambda\left(2^j\right)^{-2/N}-\tau\right)\ln\left[\frac{(a/2^j)^{2/N}}{4\pi(t-\lambda(2^j)^{-2/N}-\tau)}\right]\leqslant 2N(t-\tau)\ln\left[\frac{a^{2/N}}{4\pi(t-\tau)}\right].$$

This follows from analysis of the function $f(m) = 2Nm\ln(\frac{a^{2/N}}{4\pi m})$. The maximum of this function is reached when $m = m_0 = \frac{a^{2/N}}{4\pi e}$. However for (z,τ) in the ball $P(x, t-\lambda(2^j)^{-2/N}; \frac{a}{2^j})$ we see that

$$\frac{a^{2/N}}{4\pi e} \geqslant \frac{(a/2^j)^{2/N}}{4\pi} + \lambda\left(2^j\right)^{-2/N} \geqslant t-\tau$$

when $j \geqslant \sigma$ with σ sufficiently large since $\lambda = a^{2/N}/(2\pi)$. The function $f = f(m)$ is increasing when $m < \frac{a^{2/N}}{4\pi e}$. The proof is done since

$$t-\lambda\left(2^j\right)^{-2/N} - \tau \leqslant t-\tau. \qquad \square$$

3. Solvability condition on the nonlinear potential

From Proposition 2.1 we have the following result.

Lemma 3.1. *Let $V = V(x)$ be a L^1_{loc} function. Let P be the parabolic ball $P(x,t;a)$ for $a > 0$ and $P_j = P(x,t;\frac{a}{2^j})$ for $j \in \mathbb{N}$. Denote by χ_P and χ_{P_j} the characteristic functions of P and P_j respectively. Then for all $(y,s) \in P(x,t;a)$, there exists a positive integer σ, such that,*

$$(G *_V \chi_P)(y,s) \geqslant \phi_\sigma(y,s) \tag{3.1}$$

where

$$\phi_\sigma \equiv \sum_{j\geqslant\sigma} \eta_j \chi_{P_j}(y,s).$$

and

$$\eta_j = (ab)^{-1}2^j\left|P\left(x,t;\frac{a}{2^j}\right)\right|_V \equiv (ab)^{-1}2^j \int\limits_{P(x,t;\frac{a}{2^j})} V(y)\,dy\,ds,$$

and b is the dimensional constant given in Proposition 2.1.

Proof.

$$\begin{aligned}
(G *_V \chi_P)(y,s) &= \iint_0^s G(y,s;z,\tau)\chi_P(z,\tau)V(z)\,dz\,d\tau \\
&= \iint_0^\infty \int\limits_{G(y,s;z,\tau)>\frac{1}{r}} \chi_P(z,\tau)V(z)\,dz\,d\tau\,\frac{dr}{r^2} \qquad \text{Fubini} \\
&= \int_0^\infty \frac{|P(x,t;a)\cap P(y,s;r)|_V\,dr}{r^2} \\
&\geqslant \sum_{j=0}^\infty \int_{\frac{ab}{2^{j+1}}}^{\frac{ab}{2^j}} \frac{|P(x,t;a)\cap P(y,s;r)|_V\,dr}{r^2}
\end{aligned}$$

$$\geqslant \left(\frac{1}{ab}\right)\sum_{j=0}^{\infty} 2^j \big|P(x,t;a)\cap P\big(y,s;ab2^{-(j+1)}\big)\big|_V. \tag{3.2}$$

Here, b is from Proposition 2.1.

Fixing (y,s), let j_0 be the largest number such that $(y,s)\in P_{j_0}$. Then the containment from Proposition 2.1 holds for all $\sigma\leqslant j\leqslant j_0$ since $(y,s)\in P_j$, i.e.

$$P\left(x,t-\lambda\big(2^j\big)^{-2/N};\frac{a}{2^j}\right)\subset P(x,t;a)\cap P\left(y,s;\frac{ba}{2^{j+1}}\right). \tag{3.3}$$

Therefore,

$$\begin{aligned}
(G*_V\chi_P)(y,s) &= \iint_0^{\infty}\int_{G(y,s;z,\tau)>\frac{1}{r}} \chi_{P(x,t;r)}(z,\tau)V(z)\,dz\,d\tau\,\frac{dr}{r^2} \\
&\geqslant (ab)^{-1}\sum_{j=\sigma}^{j_0} 2^j\left|P\left(x,t-\lambda\big(2^j\big)^{-2/N};\frac{a}{2^j}\right)\right|_V \chi_{P_j}(y,s) \\
&\geqslant (ab)^{-1}\sum_{j=\sigma}^{\infty} 2^j\left|P\left(x,t-\lambda\big(2^j\big)^{-2/N};\frac{a}{2^j}\right)\right|_V \chi_{P_j}(y,s) \\
&= \sum_{j=\sigma}^{\infty}\eta_j\chi_{P(x,t;\frac{a}{2^j})}(y,s)=\phi_\sigma.
\end{aligned} \tag{3.4}$$

The last step holds as the function V is independent of time and we can take $\eta_j=(ab)^{-1}2^j|P(x,t;\frac{a}{2^j})|_V$. So we see that $G*\chi_P\geqslant\phi_\sigma$. We also used the fact that, if $j\geqslant j_0+1$, then $\chi_{P(x,t;\frac{a}{2^j})}(y,s)=0$. So we see that (3.1) holds, as desired. □

Using the above result we derive a technical lemma which is essential to our main result. This result again has its root in the paper [4] even though it is more difficult than the elliptic case.

Lemma 3.2. *Suppose $V=V(x)$ satisfies the doubling condition, i.e. there exists $D_0>0$ such that*

$$\int_{B(x,r)} V(y)\,dy\geqslant D_0\int_{B(x,2r)} V(y)\,dy$$

for all x and $r>0$.

For any $k=1,2,3,\dots$, and η_j as in the statement of the previous lemma, define

$$\phi_{\sigma,k}=\sum_{j=\sigma}^{\infty}\eta_j\chi_{P_{j+k\sigma}}.$$

Then for all $1\leqslant s<\infty$, we have

$$G*_V\phi_{\sigma,k}^s\geqslant\frac{Q^{k+1}}{s+1}\phi_{\sigma,k+1}^{s+1}. \tag{3.5}$$

Here Q depends only on the constant in the doubling condition.

Proof. For the σ as in Lemma 3.1, write $\beta_j = \sum_{i=\sigma}^{j} \eta_i$ with $\beta_{\sigma-1} \equiv 0$. Because $\eta_j = \beta_j - \beta_{j-1}$, by the mean value inequality, it is easy to see that

$$\beta_j^{s+1} - \beta_{j-1}^{s+1} \leqslant (s+1)\eta_j \beta_j^s. \tag{3.6}$$

We now make the observation that if we denote

$$\alpha_j \equiv \beta_j^s - \beta_{j-1}^s$$

then we can write

$$\phi_{\sigma,k}^s = \sum_{j=\sigma}^{\infty} \alpha_j \chi_{P_{j+k\sigma}}. \tag{3.7}$$

This is easy to see since,

$$\phi_{\sigma,k} = \sum_{j=\sigma}^{\infty} (\beta_j - \beta_{j-1}) \chi_{P_{j+k\sigma}} = \sum_{j=\sigma}^{\infty} \beta_j (\chi_{P_{j+k\sigma}} - \chi_{P_{j+k\sigma+1}}).$$

Hence further we have

$$\begin{aligned}
\phi_{\sigma,k}^s &= \left(\sum_{j=\sigma}^{\infty} \beta_j (\chi_{P_{j+k\sigma}} - \chi_{P_{j+k\sigma+1}}) \right)^s \\
&= \sum_{j=\sigma}^{\infty} \beta_j^s (\chi_{P_{j+k\sigma}} - \chi_{P_{j+k\sigma+1}}) \\
&= \sum_{j=\sigma}^{\infty} \left(\beta_j^s - \beta_{j-1}^s \right) \chi_{P_{j+k\sigma}} \\
&= \sum_{j=\sigma}^{\infty} \alpha_j \chi_{P_{j+k\sigma}}.
\end{aligned} \tag{3.8}$$

One should note that (3.8) is valid since we consider essentially disjoint sets of the type $P_{j+k\sigma} - P_{j+k\sigma+1}$.

Now observe that

$$\begin{aligned}
G *_V \phi_{\sigma,k}^s &= G *_V \left(\sum_{j=\sigma}^{\infty} \alpha_j \chi_{P_{j+k\sigma}} \right) \\
&= \sum_{j=\sigma}^{\infty} \alpha_j G *_V \chi_{P_{j+k\sigma}} \\
&\geqslant \sum_{j=\sigma}^{\infty} \alpha_j \sum_{i=j+(k+1)\sigma}^{\infty} \eta_i \chi_{P_i}.
\end{aligned}$$

Here the last step is by Lemma 3.1 via scaling. Therefore, exchanging the order of summation, we have

$$
\begin{aligned}
G *_V \phi^s_{\sigma,k} &\geqslant \sum_{i=(k+2)\sigma}^{\infty} \eta_i \left(\sum_{j=\sigma}^{i-(k+1)\sigma} \alpha_j \right) \chi_{P_i} \\
&= \sum_{i=2\sigma}^{\infty} \eta_i \beta^s_{i-(k+1)\sigma} \chi_{P_i} \quad \text{since } \alpha_j = \beta^s_j - \beta^s_{j-1}.
\end{aligned}
$$

By the doubling condition on V (on the Euclidean balls) and the assumption that V is independent of time, one can prove that V is also doubling for the heat balls. This can be done by inscribing the largest cylinder in space time in the heat balls. Therefore we have, for some $Q > 0$,

$$
\eta_i \geqslant Q^{k+1} \eta_{i-(k+1)\sigma}.
$$

Hence, it holds

$$
\begin{aligned}
G *_V \phi^s_{\sigma,k} &\geqslant Q^{k+1} \sum_{i=(k+2)\sigma}^{\infty} \eta_{i-(k+1)\sigma} \beta^s_{i-(k+1)\sigma} \chi_{P_i} \\
&\geqslant \frac{Q^{k+1}}{s+1} \sum_{i=(k+2)\sigma}^{\infty} \left(\beta^{s+1}_{i-(k+1)\sigma} - \beta^{s+1}_{i-(k+1)\sigma-1} \right) \chi_{P_i} \quad \text{from (3.6)}.
\end{aligned}
$$

This shows that

$$
\begin{aligned}
G *_V \phi^s_{\sigma,k} &= \frac{Q^{k+1}}{s+1} \sum_{i=(k+2)\sigma}^{\infty} \beta^{s+1}_{i-(k+1)\sigma} (\chi_{P_i} - \chi_{P_{i+1}}) \\
&= \frac{Q^{k+1}}{s+1} \left[\sum_{i=(k+2)\sigma}^{\infty} \beta_{i-(k+1)\sigma} (\chi_{P_i} - \chi_{P_{i+1}}) \right]^{s+1} \\
&= \frac{Q^{k+1}}{s+1} \left[\sum_{i=(k+2)\sigma}^{\infty} (\beta_{i-(k+1)\sigma} - \beta_{i-(k+1)\sigma-1}) \chi_{P_i} \right]^{s+1} \\
&\geqslant \frac{Q^{k+1}}{s+1} \left[\sum_{i=(k+2)\sigma}^{\infty} \eta_{i-(k+1)\sigma} \chi_{P_i} \right]^{s+1} = \frac{Q^{k+1}}{s+1} \left[\sum_{i=\sigma}^{\infty} \eta_i \chi_{P_{i+(k+1)\sigma}} \right]^{s+1} \\
&= \frac{Q^{k+1}}{s+1} \phi^{s+1}_{\sigma,(k+1)}. \qquad \square
\end{aligned}
$$

Now we are in a position to state and prove the main result.

Theorem 3.1. *Let $V = V(x)$ be an L^1_{loc} function satisfying the doubling condition with constant $Q > 0$, i.e. for any $r > 0$ and $x \in \mathbb{R}^N$*

$$
\int_{B(x,r)} V(y)\,dy \geqslant Q \int_{B(x,2r)} V(y)\,dy.
$$

Suppose for some nonnegative u_0 and $w \geqslant 0$, the problem

$$\begin{cases} \Delta u(x,t) - u_t(x,t) + V(x)u^p(x,t) + w(x) = 0, \\ u(x,0) = u_0(x) \end{cases} \tag{3.9}$$

has a positive solution in $\mathbb{R}^N \times (0,T]$.

Define, for any $(x_0,t_0) \in \mathbb{R}^N \times (0,T]$, the quantity

$$\Lambda(x_0,t_0) = \int G(x_0,t_0;y,0)u_0(y)\,dy + \int_0^{t_0}\!\!\int G(x_0,t_0;y,s)w(y)\,dy\,ds.$$

Then for almost all $(x_0,t_0) \in \mathbb{R}^N \times (0,T]$ and all $a > 0$, there exists $S_0 = S_0(\Lambda(x_0,t_0), p, N, Q, T) > 0$, such that

$$(G *_V \chi_P)(x_0,t_0) := \int_0^{t_0}\!\!\int G(x_0,t_0;y,s)\chi_P(y,s)V(y)\,dy\,ds < S_0 < \infty. \tag{3.10}$$

Here again $P = P(x_0,t_0,a)$ is the heat ball with vertex (x_0,t_0) and size a.

Remark 3.1. (a) One notes that the condition given in Theorem 3.1 is also almost sufficient if the potential V is bounded near infinity, i.e. the singularity of V are local. This follows from an argument in Theorem B in [5]. Indeed, for V being bounded near infinity, (3.10) easily implies that the quantity

$$K(V,h)(x_0,t_0) \equiv \int_{t_0-h}^{t_0}\!\!\int G(x_0,t_0;y,s)\chi_P(y,s)V(y)\,dy\,ds < \xi(h), \quad \text{a.e. } (x_0,t_0).$$

Here h is a positive number and $\xi > 0$ depends on h. Now if

$$\lim_{h\to 0} \sup_{(x_0,t_0)} K(V,h)(x_0,t_0) = 0.$$

Then V is in the (parabolic) Kato class. Hence (3.9) has positive solutions. For details please see [5]. Actually the existence still holds if the above limit is just sufficiently small. So the gap between our necessary condition and the sufficient condition is just the size of the above quantity $K(V,h)$.

A natural question is whether the constant S_0 in the theorem can be chosen independent of u_0 and w. The answer is trivially no since the equation has a positive solution when V is any constant.

(b) While in most applications it is sufficient to deal with nonlinear potentials V satisfying the doubling condition, at this time we do not know if the doubling condition on V is necessary for the theorem.

Proof. By standard comparison method, we can assume, without loss of generality, that $u_0 = 0$. One can also see it by observing that if u is a solution to (3.9) then the function $v = u(x,t) - \int G(x,t;y,0)u_0(y)\,dy$ satisfies

$$\begin{cases} (\Delta v - \partial_t)v(x,t) + 2^{p-1}V(x)v^p(x,t) + 2^{p-1}\Big(\int G(x,t;y,0)u_0(y)\,dy\Big)^p + w(x) \geqslant 0, \\ u(x,0) = 0. \end{cases}$$

We also consider the potential term $V = V(x)$ to be defined in all space time $\mathbb{R}^N \times (-\infty, \infty)$. The inhomogeneous term w is considered to be 0 when $t < 0$. In this manner, we can define u in all space time by setting $u(x,t) = 0$ when $t < 0$. The extended function, still called u, is clearly a solution to the equation in (3.9) in all space time.

The rest of the proof is divided into two steps. We rely heavily on the general ideas in [4]. However the proof contains a new ingredient and an added complexity to overcome the main difficulty mentioned earlier, i.e. the lack of quasi-metric conditions.

Step I. From (3.9) and Duhamel's formula we see that, for $t > 0$,

$$u(x,t) = \int_0^t\!\!\int G(x,t;y,s)V(y)u^p(y,s)\,dy\,ds + \int_0^t\!\!\int G(x,t;y,s)w(y)\,dy\,ds$$

which implies that

$$u(x,t) \geqslant \iint G(x,t;y,s)V(y)u^p(y,s)\,dy\,ds \equiv \mathcal{A}u(x,t), \tag{3.11}$$

where for convenience we have defined

$$\mathcal{A}u(x,t) = \iint G(x,t;y,s)V(y)u^p(y,s)\,dy\,ds.$$

But as the solution is positive when $t > 0$, then for any fixed $(x_0, t_0) \in \mathbb{R}^N \times (0, T]$ there exists some $\theta > 0$ and $a > 0$ such that

$$u(x,t) \geqslant \theta\chi_P(x,t) \tag{3.12}$$

where $P = P(x_0, t_0, a)$ is the heat ball with vertex at (x_0, t_0) and size a. This follows from the fact that

$$u(x,t) \geqslant \int_{\mathbf{R}^N} G(x,t;y,0)u_0(y)\,dy + \int_0^t\!\!\int_{\mathbf{R}^N} G(x,t;y,s)w(y)\,dy\,ds.$$

Recall the right-hand side of the above inequality is called $\Lambda(x,t)$ in the statement of the theorem. This is how the dependence on Λ occurs.

So we see via substitution from (3.12) and (3.11)

$$\mathcal{A}u(x,t) \geqslant \theta^p \iint G(x,t;y,s)V(y)\chi_P(y,s)\,dy\,ds.$$

Hence we have

$$\mathcal{A}u(x,t) \geqslant \theta^p \iint G(x,t;y,s)V(y)\chi_P(y,s)\,dy\,ds := \theta^p(G *_V \chi_P)(x,t). \tag{3.13}$$

Therefore, from Lemma 3.1,

$$\mathcal{A}u(x,t) \geqslant \theta^p(G *_V \chi_P)(x,t) \geqslant \theta^p\phi_\sigma(x,t). \tag{3.14}$$

Step II. Then from (3.14) we have

$$\mathcal{A}^2 u \geqslant A\big(\theta^p \phi_\sigma\big) \geqslant \theta^{p^2} G *_V \phi_\sigma^p.$$

By Lemma 3.2, this implies

$$\mathcal{A}^2 u \geqslant \theta^{p^2} \frac{Q}{p+1} \phi_{\sigma,1}^{p+1}.$$

Applying $\mathcal{A}$ on both sides of the above inequality and using Lemma 3.2 with $s = p^2 + p$, we have

$$\mathcal{A}^3 u \geqslant \theta^{p^3} \frac{Q^p}{(p+1)^p} G *_V \phi_{\sigma,1}^{p^2+p} \geqslant \theta^{p^3} \frac{Q^p}{(p+1)^p} \frac{Q^2}{p^2+p+1} \phi_{\sigma,2}^{p^2+p+1}.$$

By induction it is easy to show that

$$\mathcal{A}^n u \geqslant \theta^{p^n} \prod_{j=1}^{n-1} \big(1 + p + \cdots + p^j\big)^{-p^{n-j-1}} Q^{p^{n-2}+2p^{n-3}+\cdots+(n-2)p+n-1} \phi_{\sigma,n-1}^{1+p+\cdots+p^{n-1}}. \tag{3.15}$$

We have already seen that

$$u(x,t) \geqslant \mathcal{A}u(x,t)$$

and hence

$$u(x,t) \geqslant \mathcal{A}^n u(x,t).$$

Since u is L^1_{loc}, it must be finite a.e. Suppose $u(x_0,t_0)$ is finite. Then the above shows

$$u(x_0,t_0) \geqslant \theta^{p^n} c(n,p)\, d(n,Q) \phi_{\sigma,n-1}^{(p^n-1)/(p-1)}. \tag{3.16}$$

Here, for brevity, we have used the notations

$$c(n,p) = \prod_{j=1}^{n-1} \big(1 + p + \cdots + p^j\big)^{-p^{n-j-1}}, \qquad d(n,Q) = Q^{p^{n-2}+2p^{n-3}+\cdots+(n-2)p+n-1}.$$

Let us recall that by definition

$$\phi_{\sigma,n} = \sum_{i=\sigma}^{\infty} \eta_i \chi_{P_{i+n\sigma}}$$

with the vertex of the heat balls $\chi_{P_{i+n\sigma}}$ being at (x_0,t_0). So, for all natural numbers n,

$$\phi_{\sigma,n}(x_0,t_0) = \sum_{i=\sigma}^{\infty} \eta_i.$$

Note that the right side no longer depends on n, which is the key.

Hence (3.16) shows, after taking $n \to \infty$,

$$\phi_\sigma(x,t) = \sum_{i=\sigma}^{\infty} \eta_i \chi_{P_i} \leqslant \sum_{i=\sigma}^{\infty} \eta_i \leqslant C_1(Q,p)\theta^{-(p-1)} \tag{3.17}$$

for a.e. (x,t). Here C_1 is a constant depending only on p and Q. Note that we also used the simple fact that $C(n,p)^{\frac{p-1}{p^n-1}}$ and $d(n,Q)^{\frac{p-1}{p^n-1}}$ converge to finite constants as $n \to \infty$.

One notes that by definition we have $\phi_\sigma = \sum_{j=\sigma}^{\infty} \eta_j \chi_{P_j}$. As before if we call $\eta_j = \beta_j - \beta_{j-1}$ where $\beta_j = \sum_{i=\sigma}^{j} \eta_i$, $j \geqslant \sigma$, then we have

$$\phi_\sigma = \sum_{j=\sigma}^{\infty} \eta_j \chi_{P_j} = \sum_{j=\sigma}^{\infty} (\beta_j - \beta_{j-1})\chi_{P_j} = \sum_{j=\sigma}^{\infty} \beta_j (\chi_{P_j} - \chi_{P_{j-1}}).$$

So we see that

$$\|\phi_\sigma\|_\infty = \sup \beta_j = \sum_{i=\sigma}^{\infty} \eta_i. \tag{3.18}$$

Clearly,

$$\begin{aligned}\iint G(x_0,t_0;y,s)V(y)\chi_{P(x_0,t_0,a/2^\sigma)}(y,s)\,dy\,ds &= \sum_{j=\sigma}^{\infty} \int_{\frac{a}{2^{j+1}}}^{\frac{a}{2^j}} \frac{|P(x_0,t_0;r)|_V\,dr}{r^2} \\ &\leqslant \sum_{j=\sigma}^{\infty} \left| P\left(x_0,t_0;\frac{a}{2^j}\right)\right|_V \frac{2^j}{a} \\ &= b\sum_{j=\sigma}^{\infty} \eta_j\end{aligned}$$

where b is the constant in the definition of η_j in the first lemma of the section. This together with (3.18) shows that

$$\begin{aligned}\iint G(x_0,t_0;y,s)V(y)\chi_{P(x_0,t_0,a/2^\sigma)}(y,s)\,dy\,ds &\leqslant b\sum_{j\geqslant 0} \eta_j \\ &= b\|\phi\|_\infty \quad \text{from (3.18)} \\ &\leqslant c(N,p,Q,\theta) \equiv S_0 \quad \text{from (3.17)}.\end{aligned}$$

The proof is finished by renaming $a/2^\sigma$ by a since a is arbitrary. $\square$

For some applications, it is useful to estimate the constant S_0 in Theorem 3.1 in the following way. Generalizing from the elliptic case in [4], we introduce the following:

Definition 3.1. Given a L^1_{loc} function $f \geqslant 0$, we define $\|f\|_{\mathcal{Z}}$ to be $\inf\{\mu\}$ for which the integral equation

$$u(x,t) = \int_0^t\!\!\int G(x,t;y,s) V u^p(y,s)\,dy\,ds + \mu^{-1} f$$

has a nontrivial solution for $(x,t) \in \mathbb{R}^N \times (0,\infty)$.

Corollary 3.1. *Under the same assumptions as in Theorem* 3.1, *we have, for some $C = C(p,Q) > 0$, almost all (x_0,t_0), $0 < t_0 \leqslant T$, and all $a > 0$,*

$$C(p,Q)\left(\int_0^{a/2^\sigma} \int_{G(x_0,t_0;z,\tau) > \frac{1}{r}} \frac{V(z)}{r^2}\,dz\,d\tau\,dr\right)^{1/(p-1)} \leqslant \|\chi_{P(x_0,t_0,a)}\|_{\mathcal{Z}}.$$

Proof. Let θ be the supremum of the numbers so that the integral equation

$$u(x,t) = \int_0^t\!\!\int G(x,t;y,s) V(y) u^p(y,s)\,dy\,ds + \theta \chi_P$$

has a solution. Here again $P = P(x_0,t_0,a)$. Then $u \geqslant \theta \chi_P$. We can proceed line by line as in the proof of the theorem starting from (3.12) to reach

$$\int\!\!\int_{P(x_0,t_0,a/2^\sigma)} G(x_0,t_0;y,s) V(y)\,dy\,ds \leqslant C_1(p,Q)\theta^{p-1}.$$

By definition $\|\chi_P\|_Z = \theta^{-1}$. The corollary follows by using the usual "layer cake" integration. □

The next corollary provides a general nonexistence result in the case when the nonlinear potential V is the popular inverse square potential. In particular it contains a well-known result of Brezis and Cabré [1] (when $p = 2$) as a special case.

Corollary 3.2. *For any $h > 0$, $T > 0$ and $p > 1$, the problem*

$$\begin{cases} \Delta u(x,t) + \dfrac{h}{|x|^2} u^p(x,t) + w(x) - \partial_t u(x,t) = 0, & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \end{cases}$$

has no positive solutions. Here w and u_0 are any nonnegative functions.

Proof. It is not hard to check that the inverse square potential satisfies the doubling condition. Indeed, by direct calculation, one has, for some $Q > 0$,

$$\int_{B(x,2r)} \frac{1}{|y|^2}\,dy \leqslant Q \int_{B(x,r)} \frac{1}{|y|^2}\,dy.$$

This can be verified by considering two cases $|x| \leqslant 4r$ and $|x| \geqslant 4r$. Since $1/|y|^2$ is independent of time, this also shows that

$$\int_{P(x,t;2r)} \frac{1}{|y|^2}\, dy\, ds \leqslant C \int_{P(x,t;r)} \frac{1}{|y|^2}\, dy\, ds,$$

which is the parabolic doubling property.

By straight forward computation, it is clear that, for any $a > 0$ and $t > 0$,

$$\lim_{x \to 0} \int_0^t \int G(x, t; y, s) \chi_{P(x,t,a)} \frac{1}{|y|^2}\, dy\, ds = \infty.$$

So, the nonexistence of positive solutions follows. $\square$

It is interesting to compare this nonexistence result with another classical one by Baras and Goldstein [2]. There the authors studied the linear equation

$$\begin{cases} \Delta u(x,t) + \dfrac{h}{|x|^2} u(x,t) - \partial_t u(x,t) = 0, & (x,t) \in \mathbb{R}^N \times (0, T), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

They proved that there exists a dimensional constant h_0 so that the following holds. If $h \leqslant h_0$ then the above possesses a positive solution for nontrivial and sufficiently regular u_0. Otherwise there is no positive solution. Corollary 3.2 shows that $1/|x|^2$ as a nonlinear potential is more dominant than as a linear potential.

In this paper we are mainly dealing with the nonlinear potential term V. We hope to address the necessity question on the inhomogeneous term W, as done in [4] for the elliptic case, in the future.

Acknowledgments

We wish to thank Professors N. Kalton and I. Verbitsky for helpful communications.

References

[1] Haïm Brezis, Xavier Cabré, Some simple nonlinear PDE's without solutions, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 1 (2) (1998) 223–262 (Italian summary).

[2] P. Baras, J.A. Goldstein, Non-negative solutions of linear parabolic equations, Trans. Amer. Math. Soc. 284 (1984) 121–139.

[3] P. Baras, M. Pierre, Critére d'existence de solutions positives pour des équation semilinéaires non monotones, Ann. Inst. H. Poincaré Anal. Non Linéaire (3) (1985) 185–212.

[4] N.J. Kalton, I.E. Verbitsky, Nonlinear equations and weighted norm inequalities, Trans. Amer. Math. Soc. 351 (9) (1999) 3441–3497.

[5] Q.S. Zhang, An optimal parabolic estimate and its applications in prescribing scalar curvature on some open manifolds with *Ricci* $\geqslant 0$, Math. Ann. 316 (4) (2000) 703–731.